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Fuzzy Quantum Logic II. The Logics of Unsharp Quantum Mechanics

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A survey of the main results of the Italian group about the logics of unsharp quantum mechanics is presented. In particular partial ordered structures playing with respect to effect operators (linear bounded operators F on a Hilbert space $\mathscr{H}$ such that $\forall\psi \in \mathscr{H}$, $0 \leq \langle \psi | F\psi \rangle \leq \|\psi\|^2$) the role played by orthomodular posets with respect to orthogonal projections (corresponding to "sharp" effects) are analyzed. These structures are generally characterized by the splitting of standard orthocomplementation on projectors into two nonusual orthocomplementations (a *fuzzy-like* and an *intuitionistic-like*) giving rise to different kinds of Brouwer-Zadeh (BZ) posets: de Morgan BZ posets, BZ* posets, and BZ^3 posets. Physically relevant generalizations of ortho-pair semantics (paraconsistent, regular paraconsistent, and minimal quantum logics) are introduced and their relevance with respect to the logic of unsharp quantum mechanics are discussed.

1. THE QUANTUM LOGICAL PIONEERS: G. BIRKHOFF AND J. VON NEUMANN

The first statement about the *logic of quantum mechanics* is found in von Neumann (1932): "The relation between the *properties* of a physical system on the one hand, and the *projections* on the other, makes possible a sort of *logical calculus* with these". Some years later, Birkhoff and von Neumann (1936) went deep into this argument, asserting that:

> It is clear that an *observation* of a physical system $\mathfrak{S}$ can be described generally as a writing down of the readings from various compatible measurements. Thus if the [compatible] measurements are denoted by the symbols $\alpha_1, \ldots, \alpha_n$, then an observation of $\mathfrak{S}$ amounts to specific numbers $x_1, \ldots, x_n$ corresponding to the different α_k. It follows that the most general form of the prediction concerning $\mathfrak{S}$ is that
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> > the point $(x_1, \ldots, x_n)$ determined by actually measuring $(\alpha_1, \ldots, \alpha_n)$ will lie in a subset Δ of $(x_1, \ldots, x_n)$-space.

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Hence, if we call the $(x_1, \ldots, x_n)$-spaces associated with $\mathfrak{S}$, its *observation-spaces*, we may call the subsets of the observation-spaces with any physical system $\mathfrak{S}$, the *experimental propositions* concerning $\mathfrak{S}$.

[Footnote] One may regard a set of compatible measurements $[(\alpha_1, \ldots, \alpha_n)]$ as a single composite measurement [(which we denote by A)]—and also admit non-numerical readings—without interfering with subsequent arguments. Among conspicuous observables in quantum theory are position, momentum, energy, and (non-numerical) symmetry.

Making a little bit more formalized this BvN point of view, the set of *observables* of a physical entity is denoted by $\mathcal{O}$, the *observation space* of an observable $A \in \mathcal{O}$ is a measurable space $(\mathbb{K}_A, \mathscr{B}(\mathbb{K}_A))$ consisting of the *value-set* $\mathbb{K}_A$ and a σ-algebra $\mathscr{B}(\mathbb{K}_A)$ of *observable-subsets of* $\mathbb{K}_A$; the set of all observation spaces induced by $\mathcal{O}$ is denoted by $\mathscr{V}(\mathcal{O}) := \{(\mathbb{K}_A, \mathscr{B}(\mathbb{K}_A)): A \in \mathcal{O}\}$. An *experimental proposition* is a pair (A, Δ) consisting of the observable A and the measurable subset Δ of the value-set $\mathbb{K}_A$ associated to A; this experimental proposition corresponds to the elementary statement

"val$(A) \in \Delta$" := the value of the observable (physical magnitude) $A \in \mathcal{O}$ lies in the subset $\Delta \in \mathscr{B}(\mathbb{K}_A)$ of the observation space of A"

This agrees with the following quotation of Varadarajan (1962): "The center of the stage of the present discussion is occupied by a physical system and the *experimental propositions* that are associated with it .... if A is an *observable,* to each Borel set Δ of the real line $\mathbb{R}$ is associated the *proposition* the value of A lies in Δ". In the literature the "experimental proposition" of Birkhoff and von Neumann (1936) and Varadarajan (1962) is also called the "physical statement" (Gudder, 1970), "question" (Piron, 1972), "theoretical sentence" (Bub, 1973), and "elementary statement" (van Fraassen, 1974).

According to Birkhoff and von Neumann (1936), it is important to distinguish an experimental proposition by its mathematical representative in a suitable concrete mathematical structure; in conventional quantum mechanics, "The *mathematical representative* of any *experimental proposition* is a *closed linear subspace* of Hilbert *space*." Owing to the one-to-one correspondence between subspaces and orthogonal projections of a Hilbert space, an *experimental proposition* (A, Δ) can be mathematically realized also by an orthogonal projection $E_A(\Delta)$ of the Hilbert space $\mathscr{H}$. Let us denote by $\mathscr{M}(\mathscr{H})$ the set of all subspaces (closed linear manifolds) of $\mathscr{H}$, *by* $\mathscr{E}(\mathscr{H})$ the set of all orthogonal projections on $\mathscr{H}$, and by $\mathbb{E}: \mathscr{M}(\mathscr{H}) \mapsto \mathscr{E}(\mathscr{H})$ the mapping associating to every subspace $M \in \mathscr{M}(\mathscr{H})$ the orthogonal projection $\mathbb{E}_M \in \mathscr{E}(\mathscr{H})$ which projects onto M. An orthogonal projection is physically interpreted as an *event* produced by macroscopic apparatuses testing if the answer "yes" of a dichotomic (i.e., "yes–no") alternative occurs or does not occur. A projector, as an operator on a complex separable Hilbert space $\mathscr{H}$, is such that for any vector ψ of $\mathscr{H}$, $0 \leq \langle\psi|E_A(\Delta)\psi\rangle\|\psi\|^2$, allowing the physical interpretation of the quality

$$p(\psi, E_A(\Delta)) := \frac{\langle\psi|E_A(\Delta)\psi\rangle}{\|\psi\|^2} \in [0, 1]$$

as the *probability* of occurrence of the *event* $E_A(\Delta) \in \mathscr{E}(\mathscr{H})$ for the entity *prepared* in $\psi \in \mathscr{H} \setminus \{\underline{0}\}$. The set $\mathscr{H}_0 := \mathscr{H} \setminus \{\underline{0}\}$ of all nonzero vectors of the Hilbert space is physically interpreted as the set of all *preparations* of the physical entity described by $\mathscr{H}$.

1.1. The Empirical Semantic of Quantum Mechanics

Experimental propositions of the kind (A, Δ_1), $(B, \Delta_2), \ldots$ are *sentences* which can be put on the physical entity under examination. For instance, "the particle has passed through the slit 1," "the spin of the particle along the z direction is up," and so on. Sentences are statements which have the property of being sometimes *true* or *false* or, in some cases, also *indeterminate*. To this purpose we adopt the following *metaphysical assumption*: The truth values of elementary statements of the *sentential logic* underlying QM must be introduced making use of *only* the notions available in the concrete mathematical theory of Hilbert spaces. Precisely, we introduce two predicate signs, "*true*" T and "*false*" F, involving pairs consisting of a preparation $\psi \in \mathscr{H}_0$ and an elementary statement $r = (A, \Delta)$ according to the following definitions:

$$(\psi, r)T \quad \text{iff} \quad p(\psi, E_A(\Delta)) = 1$$

$$(\psi, r)F \quad \text{iff} \quad p(\psi, E_A(\Delta)) = 0$$

and a third predicate sign, "*indeterminate*" U: $(\psi, r)U$ iff neither $(\psi, r)T$ nor $(\psi, r)F$, i.e., iff $p(\psi, E_A(\Delta)) \neq 0, 1$. Let us notice that whether or not the statement r is "true" (resp., "false") depends on the preparation (*semantical world*) of the physical entity: an elementary statement r is "true" (resp., "false") in preparation ψ iff in this preparation the event $E_A(\Delta)$ (associated to r) occurs (resp., does not occur) with *certainty*, i.e., with probability 1 (resp., 0).

It is worth noting that the *semantical structure* assigned to the *propositional calculus* of QM does *not* allow of speaking about the fact that: "an experimental proposition (A, Δ) is *true* (or, *false*, or *indeterminate*) for a *single* individual sample (say i) of the physical system," rather than: "an experimental proposition (A, Δ) is *true* (or, *false*, or *indeterminate*) for any individual sample i of the physical entity *prepared* according to a well-defined procedure ψ." Indeed, it is important to distinguish a *single test* of a question (A, Δ), which involves an individual sample i of the physical entity yielding one of the two possible answers "yes" or "no," from an *elementary experiment* of the same question, which involves a preparation ψ of individual samples yielding one of the three possible values "true" (if the result "yes" is certain in the involved preparation, i.e., probability 1), "false" (if the result "no" is certain in the preparation, i.e., probability 0), and "indeterminate" (in all remaining cases).

For any Hilbert space projector $E_A(\Delta) \in \mathscr{E}(\mathscr{H})$ the *certainly-yes* and the *certainly-no subspaces* are defined respectively as follows:

$$M_1(E_A(\Delta)) := \{\psi \in \mathscr{H}: E_A(\Delta)\psi = \psi\} = \{\psi \in \mathscr{H}: \langle \psi | E_A(\Delta)\psi \rangle = \|\psi\|^2\}$$

$$M_0(E_A(\Delta)) := \{\varphi \in \mathscr{H}: E_A(\Delta)\varphi = \underline{0}\} = \{\varphi \in \mathscr{H}: \langle \varphi | E_A(\Delta)\varphi \rangle = 0\}$$

These certainty subspaces are identified with the *certainly-yes* and *certainly-no domains* consisting of all preparation procedures with respect to which the experimental proposition (A, Δ), tested by the corresponding event $E_A(\Delta)$, is true and false, respectively:

$$S_1(E_A(\Delta)) := \{\psi \in \mathscr{H}_0: P(\psi, E_A(\Delta)) = 1\} \equiv M_1(E_A(\Delta))$$

$$S_0(E_A(\Delta)) := \{\varphi \in \mathscr{H}: P(\varphi, E_A(\Delta)) = 0\} \equiv M_0(E_A(\Delta))$$

Summarizing the Hilbert space approach to quantum logic, we have the scheme:

Experimental proposition

$$(A, \Delta) \in \mathscr{L}_{QM}$$

$$\downarrow$$

$$[(A, \Delta)]_{\simeq} \in \mathscr{L}_{\simeq} \quad \leftrightarrow \quad E_A(\Delta) \in \mathscr{E}(\mathscr{H}) \quad \leftrightarrow \quad M_1(E_A(\Delta)) \in \mathscr{M}(\mathscr{H})$$

Physical quality (property)	Projection (event)	Subspace (proposition)

Quoting Birkhoff and von Neumann (1936): "One can interpret as a *physical quality* the set of all experimental propositions (i.e., $[(A, \Delta)]_{\simeq}$) logically equivalent to a given experimental proposition (i.e., (A, Δ)) [with respect to the equivalence relation $(B, \tilde{\Delta}) \simeq (A, \Delta)$ iff $E_B(\tilde{\Delta}) = E_A(\Delta)$]. Thus subspaces of Hilbert space correspond one-many to experimental propositions, but one-to-one to physical qualities in this sense."

In conclusion, according to BvN approach to quantum logic, any experimental proposition (question which can be put to the physical entity) (A, Δ), or the associated property (physical quality of the entity) $[(A, \Delta)]_{\simeq}$, is tested by the event (projector of the mathematical description) $E_A(\Delta)$ giving rise to two propositions (subspaces of Hilbert space) $M_1(E_A(\Delta))$ and

$M_0(E_A(\Delta)) = M_1(E_A(\Delta))^{\perp}$ identified with the set of all preparation procedures yielding with certainty the answers "yes" and "no" to the involved question, respectively.

1.2. The Sentential Logic of Quantum Mechanics

Making use of the connectives "and," "or," "not," it is possible to construct on the basis of *elementary sentences* $r = (A, \Delta_1)$, $s = (B, \Delta_2)$, ... the *complex sentences* "r & s," "$r \ \underline{o} \ s$," "$\neg r$" of the *sentential logic* $\mathscr{L}_{\mathrm{QM}}$ of quantum mechanics. This sentential logic has a structure

$$\mathbf{L}_{\mathrm{QM}} = \langle \mathscr{L}_{\mathrm{QM}}, \&, \underline{o}, \neg, O, I \rangle$$

consisting of the set $\mathscr{L}_{\mathrm{QM}}$ of all *sentences* generated by (and so closed with respect to) the connectives "and" &, "or" $\underline{o}$, "not" $\neg$, starting from the set of all elementary sentences. For instance, the sentence "the particle has passed through the slit 1 *or* through the slit 2," can be described by a complex sentence of the form "$(Q, \Delta_1) \ \underline{o} \ (Q, \Delta_2)$." As another example, let us consider a system consisting of two separated particles $(1 + 2)$; the statement "the spin of particle 1 along the z direction is "up" in region Δ_1 *and* the spin of particle 2 along the x direction is "down" in region Δ_2" is described by a complex sentence of the form "$(S_z^{(1)}, \Delta_1^{\uparrow})$ & $(S_x^{(2)}, \Delta_2^{\downarrow})$." Among these elementary sentences we have two privileged sentences: the *absurd* one O, false in every semantical world, and the *certain* one I, true in every semantical world.

In agreement with Birkhoff and von Neumann (1936): "One can reasonably expect to find a *calculus of propositions* [of quantum mechanics] which is formally indistinguishable from the *calculus of linear subspaces* [of a Hilbert space] with respect to *set products* $(M_1 \cap M_2)$, *linear sums* $(M_1 + M_2)$, and *orthogonal complements* $(M^{\perp})$—and resembles the usual calculus of propositions with respect to *and* $(r_1 \& r_2)$, *or* $(r_1 \ \underline{o} \ r_2)$, and *not* $(\neg r)$."

To be precise, the set of all subspaces of $\mathscr{H}$ has a structure

$$\mathbf{M}(\mathscr{H}) = \langle \mathscr{M}(\mathscr{H}), \wedge, \vee, {}^{\perp}, \{\underline{0}\}, \mathscr{H} \rangle$$

of a complete lattice with respect to the usual set-theoretic inclusion $\subseteq$; the set product is the lattice meet $(M_1 \wedge M_2 = M_1 \cap M_2)$ and the linear sum is the lattice join $(M_1 \vee M_2 = \overline{M_1 + M_2})$. This lattice is bounded by the trivial subspaces $\{\underline{0}\}$ (the minimum element) and $\mathscr{H}$ (the maximum element) and is orthocomplemented by the mapping $^{\perp}$ associating to any subspace $M \in \mathscr{M}(\mathscr{H})$ the corresponding annihilator $M^{\perp} = \{\varphi \in \mathscr{H} : \forall \psi \in M, \langle \phi \mid \psi \rangle = 0\} \in \mathscr{M}(\mathscr{H})$. The lattice $\mathscr{M}(\mathscr{H})$ is orthomodular (or weakly distributive).

According to Birkhoff and von Neumann (1936), "The set-theoretical product [i.e., $M_1(E_A(\Delta_1)) \cap M_1(E_B(\Delta_2))$] of any two mathematical representatives [i.e., $M_1(E_A(\Delta_1))$ and $M_1(E_B(\Delta_2)) \in \mathcal{M}(\mathcal{H})$] of experimental propositions [i.e., (A, Δ_1) and $(B, \Delta_2) \in \mathcal{L}_{\mathrm{QM}}$] concerning a quantum-mechanical system, is itself the mathematical representative of a proposition [i.e., (A, Δ_1) & (B, Δ_2)]." Moreover, "the representative of the negative [i.e., $\neg(A, \Delta)$] of any experimental proposition [i.e., (A, Δ)] is the orthogonal complement [i.e., $M_1(E_A(\Delta_1))^{\perp}$] of the mathematical representative of the proposition itself [i.e., $M_1(E_A(\Delta_1))$]." All this can be formalized introducing a mapping called the *simple valuation-mapping*,

$$v_s: \quad \mathcal{L}_{\mathrm{QM}} \mapsto \mathcal{M}(\mathcal{H})$$

associating to any elementary sentence (A, Δ), tested by the event $E_A(\Delta)$, the simple proposition $M_1(E_A(\Delta))$ (i.e., the set of all semantical world in which the sentence is true) in such a way that the following holds:

$$v_s((A, \Delta_1) \,\&\, (B, \Delta_2)) = M_1(E_A(\Delta_1)) \wedge M_1(E_B(\Delta_2))$$

$$v_s((A, \Delta_1) \,\underline{o}\, (B, \Delta_2)) = M_1(E_A(\Delta_1)) \vee M_1(E_B(\Delta_2))$$

$$v_s(\neg(A, \Delta)) = M_1(E_A(\Delta))^{\perp}$$

$$v_s(O) = \{\underline{0}\}$$

$$v_s(I) = \mathcal{H}$$

Let us notice that according to Birkhoff and von Neumann (1936), "By the negative [i.e., $\neg(A, \Delta)$] of any experimental proposition [i.e., (A, Δ)] ([characterized by the] subset Δ of [the] observation-space [of observable A]) is meant the experimental proposition [i.e., (A, Δ^c)] corresponding to the set-complement [i.e., Δ^c] of Δ in the same observation-space." All this can be generalized, for any fixed observable $A \in \mathcal{O}$, by the following formalization, whatever be $\Delta, \Delta_1, \Delta_2 \in \mathcal{B}(\mathbb{K}_A)$:

$$(A, \Delta_1) \,\&\, (A, \Delta_2) := (A, \Delta_1 \cap \Delta_2)$$

$$(A, \Delta_1) \,\underline{o}\, (A, \Delta_2) := (A, \Delta_1 \cup \Delta_2)$$

$$\neg(A, \Delta) := (A, \Delta^c)$$

2. FROM EVENTS TO EFFECTS: NONUSUAL ORTHOCOMPLEMENTATIONS IN THE POSET OF EFFECTS

We have seen that the set of all propositions (subspaces) of a Hilbert space is an orthomodular orthocomplemented complete lattice which is in a one-to-one correspondence with the set of all events (orthogonal projections) on $\mathcal{H}$: the orthogonal projection $\mathbb{E}_M$ is the mathematical representative of the event which tests on preparations of the physical entity if proposition M is either "true" or "false," or "indeterminate." The set of all *events* is an orthomodular orthocomplemented complete lattice, too:

$$\mathbf{E}(\mathscr{H}) = \langle \mathscr{E}(\mathscr{H}), \wedge, \vee, ', \mathbb{O}, \mathbb{1} \rangle$$

with respect to the partial order relation

$$E_1 \leq E_2 \qquad \text{iff} \quad \forall \varphi \in \mathscr{H}, \quad \langle \varphi \mid E_1 \varphi \rangle \leq \langle \varphi \mid E_2 \varphi \rangle \tag{2.1}$$

The null operator, $\forall \varphi \in \mathscr{H}$, $\mathbb{O}(\varphi) = \underline{0}$, and the identity operator, $\forall \varphi \in \mathscr{H}$, $\mathbb{1}(\varphi) = \varphi$, are both orthogonal projections describing the *absurd* and the *certain events*, respectively. The orthocomplementation mapping $': \mathscr{E}(\mathscr{H}) \mapsto \mathscr{E}(\mathscr{H})$ associates to any $E \in \mathscr{E}(\mathscr{H})$ the orthogonal projection

$$E' := \mathbb{1} - E = \mathbb{E}_{\ker(E)} \tag{2.2}$$

and is such that the following hold, whatever be $E, E_1, E_2 \in \mathscr{E}(\mathscr{H})$:

(oc-1) $E = E''$.
(oc-2a) $E_1' \wedge E_2' = (E_1 \vee E_2)'$.
(oc-2b) $E_1' \vee E_2' = (E_1 \wedge E_2)'$.
(oc-3a) $E \wedge E' = \mathbb{O}$.
(oc-3b) $E \vee E' = \mathbb{1}$.

Condition (oc-1) is the algebraic counterpart of the "*double negation*" law, (oc-2a,b) of the de Morgan's laws [which, under condition (oc-1), are mutually equivalent; moreover they are equivalent to the "*contraposition*" law: let $E_1, E_2 \in \mathscr{E}(\mathscr{H})$; then $E_1 \leq E_2$ implies $E_2' \leq E_1'$], (oc-3a) of the "*noncontradiction*," and (oc-3b) of the *excluded middle* law of negation.

Since an operator $E \in \mathscr{E}(\mathscr{H})$ is an *event* (i.e., an orthogonal projector) iff it is

- linear, bounded, self-adjoint, and idempotent ($E^2 = E$)

one can enlarge the class of all events to the set of all *effects* $\mathscr{F}(\mathscr{H})$: An operator $F \in \mathscr{F}(\mathscr{H})$ is an *effect* (i.e., a generalized projector) iff it is

- linear, bounded, self-adjoint, positive, and absorbing ($0 \leq \langle \psi \mid F\psi \rangle \leq \|\psi\|^2$)

Trivially, every event is an effect [$\mathscr{E}(\mathscr{H}) \subseteq \mathscr{F}(\mathscr{H})$], but there exist effects which are not events [$\frac{1}{2}\mathbb{1} \in \mathscr{F}(\mathscr{H}) \setminus \mathscr{E}(\mathscr{H})$]. The physical interpretation of events as yes–no alternatives produced by macroscopic apparatuses in single tests with individual samples of the physical entity can be extended to effects. According to Kraus (1983), "Another empirical fact is the existence of so called measuring instruments which are capable of undergoing macroscopically observable changes (*effects*) due to their interaction, with single microsystems (...) One usually defines the result of a *single* *measurement* to be 'yes' if the effect occurs and 'no' if the effect does not occur." The *probability* of occurrence of the *effect* F for the entity prepared in ψ is the number

$$p(\psi, F) := \frac{\langle\psi \mid F\psi\rangle}{\|\psi\|^2} \in [0, 1]$$

For *generalized quantum mechanics* (GQM) based on the Hilbert space $\mathscr{H}$ one generally means the triple $\langle\mathscr{H}_0, \mathscr{F}(\mathscr{H}), p\rangle$, which contains, as a particular substructure, standard quantum mechanics (QM) described by the triple $\langle\mathscr{H}_0, \mathscr{E}(\mathscr{H}), p\rangle$.

The set of all *effects* of a Hilbert space has a structure

$$\mathbf{F}(\mathscr{H}) = \langle\mathscr{F}(\mathscr{H}), \leq, ', {}^{\sim}, \mathbb{0}, \mathbb{1}\rangle$$

of a poset (which is not a lattice) with respect to the *phenomenological* partial order relation $[\forall F_1, F_2 \in \mathscr{F}(\mathscr{H})]$

$$F_1 \leq F_2 \quad \text{iff} \quad \forall\varphi\in\mathscr{H}, \quad \langle\varphi \mid F_1\varphi\rangle \leq \langle\varphi \mid F_2\varphi\rangle$$

The restriction of this partial order relation to the set of all events $\mathscr{E}(\mathscr{H})$ is just the partial order relation (2.1). The standard orthocomplementation (2.2) is split into two kinds of nonusual orthocomplementations:

1. The mapping $': \mathscr{F}(\mathscr{H}) \mapsto \mathscr{F}(\mathscr{H})$ defined as

$$F' := \mathbb{1} - F$$

which satisfies the conditions, whatever be $F, G \in \mathscr{F}(\mathscr{H})$:

(doc-1) $F = F''$.
(doc-2) $F \leq G$ implies $G' \leq F'$.
(re) $F \leq F'$ and $G' \leq G$ imply $F \leq G$.

2. The mapping $^{\sim}: \mathscr{F}(\mathscr{H}) \mapsto \mathscr{F}(\mathscr{H})$ defined as

$$F^{\sim} = \mathbb{E}_{\ker(F)}$$

which satisfies the conditions, whatever be $F, G \in \mathscr{F}(\mathscr{H})$:

(woc-1) $F \leq F^{\sim\sim}$.
(woc-2) $F \leq G$ implies $G^{\sim} \leq F^{\sim}$.
(woc-3) $F \wedge F^{\sim} = 0$.

These two "orthocomplementations" are linked by the interconnection rule $\forall F \in \mathscr{F}(\mathscr{H})$:

(in) $F^{\sim\prime} = F^{\sim\sim}$.

Note that the following rules of standard orthocomplementation do *not* hold: noncontradiction $(\forall F, F \wedge F' = \mathbb{0})$ and excluded middle

$(\forall F, F \vee F' = \mathbb{1})$ for orthocomplementation 1, and excluded middle $(\forall F. F \vee F^{\sim} = \mathbb{1})$ for orthocomplementation 2. In particular,

$$\left(\frac{1}{2}\mathbb{1}\right) \wedge \left(\frac{1}{2}\mathbb{1}\right)' = \left(\frac{1}{2}\mathbb{1}\right) \vee \left(\frac{1}{2}\mathbb{1}\right)' = \left(\frac{1}{2}\mathbb{1}\right) \neq \{\mathbb{0}, \mathbb{1}\}$$

$$\left(\frac{1}{2}\mathbb{1}\right) \vee \left(\frac{1}{2}\mathbb{1}\right)^{\sim} = \left(\frac{1}{2}\mathbb{1}\right) \neq \{\mathbb{0}, \mathbb{1}\}$$

Moreover, orthocomplementation 2 does not satisfy the strong double negation law $(\forall F, F = F^{\sim})$, since

$$\left(\frac{1}{2}\mathbb{1}\right) < \left(\frac{1}{2}\mathbb{1}\right)^{\sim\sim} = \mathbb{1}$$

3. BROUWER–ZADEH (BZ) POSET

A BZ structure is a poset in which two nonusual orthocomplementation mappings are introduced in order to define an abstract environment in which some mathematical properties of effects on Hilbert space can be studied.

A *pre-BZ poset* is a poset

$$\langle \Sigma, 0, \leq, ', {}^{\sim} \rangle$$

with respect to a partial order relation $\leq$, lower bounded by 0 (hence 0 is the *least* element of Σ) and equipped with:

(BZ-1) The *Zadeh* (or *fuzzy-like*) orthocomplementation mapping $': \Sigma \mapsto \Sigma$ for which the following hold whatever be $a, b \in \Sigma$:

(doc-1) $a = a''$.
(doc-2) $a \leq b$ implies $b' \leq a'$.

(BZ-2) The *Brouwer* (or *intuitionistic-like*) orthocomplementation mapping ${}^{\sim}: \Sigma \mapsto \Sigma$ for which the following hold whatever be $a, b \in \Sigma$:

(woc-1) $a \leq a^{\sim\sim}$.
(woc-2) $a \leq b$ implies $b^{\sim} \leq a^{\sim}$.

(BZ-3) The two nonusual orthocomplementations must satisfy the interconnection rule whatever be $a \in \Sigma$:

(in) $a^{\sim\prime} = a^{\sim\sim}$

The *greatest* element of Σ exists and it is $1 := 0' = 0^{\sim}$ $(\forall a \in \Sigma, a \leq 1)$. Trivially, $a^{\sim} \leq a'$. Elements $h \in \Sigma$ for which condition $h = h'$ holds, if they exist, are called *half* elements.

A *BZ-poset* is a pre-BZ poset in which the further conditions are satisfied:

(re) $a \le a'$ and $b' \le b$ imply $a \le b$.
(woc-3) $a \wedge a^{\sim} = 0$.

Under condition (re) there exists at most a *half* element which, if it exists, will be denoted by $(1/2) \in \Sigma$. If Σ is a BZ lattice, condition (re) is equivalent to the following *Kleene* condition:

(KL) $\forall a, b \in \Sigma,\ a \wedge a' \le b \vee b'$.

The elements of $\Sigma_e := \{f \in \Sigma : f = f^{\sim\sim}\}$ are the *exact* (or *sharp*) elements of Σ, whereas the elements of $\Sigma \backslash \Sigma_e$ are the *fuzzy* (or *unsharp*) elements (Cattaneo and Nisticò, 1989). Of course, $0, 1 \in \Sigma_e$. Moreover, for every $f \in \Sigma_e, f' = f^{\sim} \in \Sigma_e$ and the mapping $f \in \Sigma_e \mapsto f' = f^{\sim} \in \Sigma_e$ is a standard orthocomplementation in $\langle \Sigma_e, 0. \le \rangle$, so that $\langle \Sigma_e, 0, \le, ' \rangle$ is an orthocomplemented bounded poset. (If Σ is a lattice, then Σ_e is a sublattice of Σ in which $f \vee_e g = f \vee g$ and $f \wedge_e g = f \wedge g$). The set $\mathscr{F}(\mathscr{H})$ of all effects (generalized projections) of a Hilbert space $\mathscr{H}$ is a BZ poset whose family $\mathscr{F}_e(\mathscr{H})$ of exact (sharp) elements is just the orthomodular lattice $\mathscr{E}(\mathscr{H})$ of all orthogonal projections on $\mathscr{H}$; fuzzy (unsharp) effects are thus generalized projections which are not orthogonal.

For every $a \in \Sigma$, one can construct two exact elements combining the two orthocomplementation mappings $a^{\sim\prime} \in \Sigma_e$ and $a^{\prime\sim} \in \Sigma_e$ allowing one to introduce the *necessity mapping* $a \in \Sigma \mapsto \nu(a) := a^{\prime\sim} \in \Sigma_e$ and the *possibility mapping* $a \in \Sigma \mapsto \mu(a) := a^{\sim\prime} \in \Sigma_e$. In particular, the necessity of an element "implies" the element itself, which in its turn "implies" the corresponding possibility $[\nu(a) \le a \le \mu(a)]$; necessity and possibility are both idempotent $[\nu(\nu(a)) = \nu(a)$ and $\mu(\mu(a)) = \mu(a)]$ and are linked by the expected interconnection rules between modal-like operators $\mu(a) = \nu(a')'$ [possibility = not-necessity-not] and $\nu(a) = \mu(a')'$ [necessity = not-possibility-not]. An interconnection rule involving intuitionistic-like orthocomplementation and modal-like operators can be stated: $\nu(a^{\sim}) = \mu(a)^{\sim}$ [in general, $\mu(a^{\sim}) \ne \nu(a)^{\sim}$]. Operators ν and μ act on the exact elements of Σ_e as the identity operators, i.e. $\forall f \in \Sigma_e, \nu(f) = \mu(f) = f$ [hence $\Sigma_e = \nu(\Sigma) = \mu(\Sigma)$]. Making use of the two unusual orthocomplementations, it is possible to introduce the *weak anti-intuitionistic orthocomplementation* $a \in \Sigma \mapsto a^{\flat} := a^{\prime\sim\prime} \in \Sigma_e$, which satisfies the following conditions:

(aoc-1) $a^{\flat\flat} \le a$.
(aoc-2) $a \le b$ implies $b^{\flat} \le a^{\flat}$.
(aoc-3) $a \vee a^{\flat} = 1$.

Trivially, for every $a \in \Sigma$, $a^{\sim} \le a' \le a^{\flat}$; moreover, the following equalities holds: $a^\flat = \nu(a)' = \nu(a)^\sim = \nu(a)^\flat$ ($a^\flat$ is the *nonnecessity* of a) and $a^\sim = \mu(a)' = \mu(a)^\sim = \mu(a)^\flat$ ($a^\sim$ is the *impossibility* of a).

We now introduce another intersting BZ structure.

Definition 3.1. A *pre-BZ** (resp., *BZ**) *poset* is any pre-BZ (resp., BZ) poset which satisfies the condition:

$$\forall a, b \in \Sigma, \quad a \le b \quad \text{iff } \nu(a) \le \nu(b) \text{ and } \mu(a) \le \mu(b).$$

The *necessity* of an effect $F \in \mathscr{F}(\mathscr{H})$ is given by $\nu(F) = \mathbb{E}_{\ker(\mathbb{1} - F)}$ and the *possibility* by $\mu(F) = \mathbb{E}'_{\ker(F)}$. The BZ poset of all effects is not a BZ* poset, since $\mathbb{E}_{\ker(\mathbb{1} - F)} \le \mathbb{E}_{\ker(\mathbb{1} - G)}$ and $\mathbb{E}_{\ker(G)} \le \mathbb{E}_{\ker(F)}$ [i.e., $\ker(\mathbb{1} - F) \subseteq \ker(\mathbb{1} - G)$ and $\ker(G) \subseteq \ker(F)$] in general do not imply $F = G$. Note that to every effect F it is possible to associate the *modality pair* $(\mathbb{E}_{\ker(\mathbb{1} - F)}, \mathbb{E}_{\ker(F)})$, which is in a one-to-one correspondence with the orthopair of subspaces $(\ker(\mathbb{1} - F), \ker(F))$.

The fuzzy orthocomplementation satisfies both the generalized de Morgan laws:

(dM-1d) Let $a, b \in \Sigma$; if $a \vee b$ exists in Σ, then $a' \wedge b'$ exists in Σ, too, and $a' \wedge b' = (a \vee b)'$.

(dM-2d) Let $a, b \in \Sigma$; if $a \wedge b$ exists in Σ, then $a' \vee b'$ exists in Σ, too, and $a' \vee b' = (a \wedge b)'$.

The intuitionistic orthocomplementation generally satisfies only the first generalized de Morgan law.

(dM-1w) Let $a, b \in \Sigma$; if $a \vee b$ exists in Σ, then $a^\sim \wedge b^\sim$ exists in Σ, too, and $a^\sim \wedge b^\sim = (a \vee b)^\sim$.

We conclude this section by introducing two other interesting BZ structures, which will be very interesting in the sequel in order to individuate some other algebraic counterpart of fuzzy-intuitionistic logics of quantum mechanics.

Definition 3.2. A *pre-de Morgan BZ* (resp., *de Morgan*) *poset* is a pre-BZ (resp., BZ) poset in which the dual of the de Morgan law for the intuitionistic-like orthocomplementation holds:

(dM-2w) Let $a, b \in \Sigma$; if $a \wedge b$ exists in Σ, then $a^\sim \vee b^\sim$ exist in Σ, too, and $a^\sim \vee b^\sim = (a \wedge b)^\sim$.

A *three-valued pre-BZ* (resp., *three-valued BZ*) *poset* or *pre-BZ^3* (resp., *BZ^3*) *poset* is a de Morgan pre-BZ* (resp., BZ*) poset.

In a finite-dimensional Hilbert space the set of all effects is always a de Morgan BZ poset, which is not a BZ^3 poset (since the set of all effects is never a BZ* poset); but it is possible to single out a nontrivial (i.e., strictly enclosing the BZ poset of orthogonal projections) BZ* poset consisting of effects which is not de Morgan (Cattaneo and Giuntini, 1993).

4. BZ³ STRUCTURES IN HILBERT SPACE QUANTUM MECHANICS INDUCED BY PHYSICAL SEPARATION RELATIONS

The notions of *certainly-yes* and *certainly-no subspaces* introduced on the set of orthogonal projections can be extended to the set of generalized projections: let $F \in \mathscr{F}(\mathscr{H})$; then

$$M_1(F) = \{\psi \in \mathscr{H} : F\psi = \psi\} = \{\psi \in \mathscr{H} : \langle \psi \mid F\psi \rangle = \|\psi\|^2\}$$

and

$$M_0(F) = \{\varphi \in \mathscr{H} : F\varphi = \underline{0}\} = \{\varphi \in \mathscr{H} : \langle \varphi \mid F\varphi \rangle = 0\}$$

Therefore, we can introduce a mapping, called the *extensional mapping*, associating to any effect of quantum mechanics the pair consisting of the certainly-yes and the certainly-no subspaces; formally:

$$ext\colon \quad \mathscr{F}(\mathscr{H}) \mapsto \mathscr{M}(\mathscr{H}) \times \mathscr{M}(\mathscr{H}), \quad F \to (M_1(F), M_0(F))$$

Note that $M_1(F)\,(\perp)\,M_0(F)$, i.e., for any $\psi \in M_1(F)$ and any $\varphi \in M_0(F)$, $\langle \psi \mid \varphi \rangle = 0$. Taking into account all these results, in the sequel we are interested in considering as a *proposition* any pair $(M_1, M_0) \in \mathscr{M}(\mathscr{H}) \times \mathscr{M}(\mathscr{H})$ of mutually orthogonal subspaces of the Hilbert space, rather than a single subspace. Subspace M_1 (resp., M_0) is the certainly-yes (resp., no) domain of the proposition, i.e., the set of all *semantical worlds* in which the proposition is true (resp., false). The usual orthogonality relation on nonzero vectors of the Hilbert space $\mathscr{H}$ is a *preclusivity* (i.e., irreflexive and symmetrical) relation and then $\langle \mathscr{H}_0, \perp \rangle$ is an *orthoframe* (Dishkant, 1972; Dalla Chiara, 1986; Dalla Chiara and Giuntini, 1989; Cattaneo, 1992). We recall that this orthogonality relation can be equivalently stated as the following binary relation of *physical separability* of preparations by effects:

$$\psi \perp \varphi \quad \text{if} \quad \exists F \in \Sigma(\mathscr{H})\colon \quad P(\psi, F) = 1 \quad \text{and} \quad P(\varphi, F) = 0 \qquad (4.1)$$

In this section we introduce a generalization of this relation of physical separability according to the following:

Definition 4.1. On the set $\mathscr{H}_0$ of all nonzero vectors of the Hilbert space $\mathscr{H}$: For any fixed $\epsilon \in [0, \frac{1}{2}]$ the ϵ-separation relation $\perp_\epsilon$ is defined as follows:

$$\psi_1 \perp_\epsilon \psi_2 \quad \text{if} \quad \exists F \in \Sigma(\mathscr{H})\colon \quad P(\psi_1, F) \geq 1 - \epsilon \quad \text{and} \quad P(\psi_2, F) \leq \epsilon \quad (4.2a)$$

This separation relation satisfies the two conditions:

(se-1a) $\psi_1 \perp_\epsilon \psi_2$ implies $\psi_2 \perp_\epsilon \psi_1$ (symmetric).

(se-2a) $\psi_1 \perp_\epsilon \psi_2$ implies $\psi_1 \neq \psi_2$ (irreflexive).

For any fixed $\epsilon \in [\frac{1}{2}, 1]$ the ϵ-separation relation $\perp_\epsilon$ is defined as follows:

$$\psi_1 \perp_\epsilon \psi_2 \quad \text{iff} \quad \exists F \in \Sigma(\mathscr{H}) \backslash \{\lambda \mathbb{I}: \lambda \in [\epsilon, 1-\epsilon]\}: \quad P(\psi_1, F) \geq 1-\epsilon \text{ and } P(\psi_2, F) \leq \epsilon \tag{4.2b}$$

This separation relation satisfies the two conditions (Cattaneo and Giuntini, 1993):

(se-1b) $\psi_1 \perp_\epsilon \psi_2$ implies $\psi_2 \perp_\epsilon \psi_1$ (symmetric).

(se-1b) $\psi_1 \perp_\epsilon \psi_1$ and $\psi_2 \perp_\epsilon \psi_2$ imply $\psi_1 \perp_\epsilon \psi_2$ (regular).

Therefore, $\forall \epsilon \in [\frac{1}{2}, 1]$, the relation $\perp_\epsilon$ of physical separation (of preparations by effects) is symmetric and regular, i.e., a *regular paraconsistency relation*, whereas $\forall \epsilon \in [0, \frac{1}{2}]$, the relation $\perp_\epsilon$ is symmetric and irreflexive (and so necessarily regular), i.e., a *preclusivity* (or *orthogonality*) *relation*. As mentioned by Dalla Chiara and Giuntini (1989), very important logics can be semantically characterized by *Kripke frames* $\mathbf{F} = \langle X, \# \rangle$, where X is a nonempty set, the *carrier space* of $\mathbf{F}$, and $\#$ is a *binary relation* on X (i.e., $\# \subseteq X \times X$) which satisfies suitable conditions. In particular, we can quote *paraconsistent quantum logics* (PQL) for which $\#$ is symmetric; *regular* PQL (RPQL) for which $\#$ is symmetric and regular; and *minimal quantum logics* (MQL) for which $\#$ is symmetric and irreflexive.

Let $\langle X, \# \rangle$ be a Kripke frame; then for any $A \subseteq X$ let us define

$$A^{\#} := \{b \in X: \forall a \in A, b \# a\}$$

and let us put $A^{\#\#} = (A^{\#})^{\#}$ and so on. A set $A \subseteq X$ is said to be an *exact* (or *#-closed*) set iff $A = A^{\#\#}$. The set of all exact sets of the frame $(X, \#)$ is denoted by $\mathscr{M}(X, \#) := \{A \subseteq X: A = A^{\#\#}\}$. The trivial subsets $X^{\#}$ and X are elements of $\mathscr{M}(X, \#)$.

Theorem 4.1. Let $\langle X, \# \rangle$ be a Kripke frame; then the structure

$$\langle \mathscr{M}(X, \#), \subseteq, X^{\#}, X \rangle$$

of all #-closed subsets of X is a complete lattice with respect to the partial ordering $\subseteq$ of set-theoretic inclusion; in particular, for any family $\{A_i\}$ of #-closed sets from $\mathscr{M}(X, \#)$:

(i) The greatest lower bound (g.l.b.), written $\bigwedge A_j$, exists and turns out to be the set-theoretic intersection $\bigwedge A_j = \bigcap A_j$.

(ii) The least upper bound (l.u.b.), written $\bigvee A_j$, exists and $\bigvee A_j = (\bigcup A_j)^{\#\#}$, which contains, and in general does not coincide with, the set-theoretic union.

The behavior of the mapping

$$^{\#}: \quad \mathcal{M}(X, \#) \mapsto \mathcal{M}(X, \#), \quad A \to A^{\#}$$

can be classified according to the following cases:

(PQL) If $\#$ is a paraconsistency relation, then $^{\#}$ is a degenerate (fuzzy-like) orthocomplementation for $\mathcal{M}(X, \#)$, i.e. the following hold, whatever be $A, B \in \mathcal{M}(X, \#)$:

(doc-i) $A = A^{\#\#}$.
(doc-iia) $B^{\#} \wedge A^{\#} = (A \vee B)^{\#}$.
(doc-iib) $B^{\#} \vee A^{\#} = (A \wedge B)^{\#}$.

(RPQL) If $\#$ is a regular paraconsistency relation then $^{\#}$ is a Kleene orthocomplementation for $\mathcal{M}(X, \#)$, i.e. besides (doc-i) and (doc-iia, b), the following holds, whatever be $A, B \in \mathcal{M}(X, \#)$:

(Kl) $A \wedge A^{\#} \subseteq B^{\#} \vee B$.

(MQL) If $\#$ is a preclusivity relation, then $^{\#}$ is a standard orthocomplementation for $\mathcal{M}(X, \#)$, i.e., besides (doc-i), (doc-ii), and (Kl), the following hold, for every $A \in \mathcal{M}(X, \#)$:

(oc-iii) $A \wedge A^{\#} = A \cap A^{\#} = \varnothing$ and $A \vee A^{\#} = X$.
(a) $X^{\#} = \varnothing$.

In the complete lattice structure with nonusual orthocomplementation $\langle \mathcal{M}(X, \#), \wedge, \vee, ^{\#}, X^{\#}, X \rangle$, elements of $\mathcal{M}(X, \#)$ are also called *simple propositions*. Let $(X, \#)$ be a Kripke frame; a pair $p = (A_T, A_F)$ of subsets of X is said to be *$\#$-consistent* iff $A_T\ (\#)\ A_F$, i.e., $\forall a \in A_T, \forall b \in A_F, a \# b$.

Definition 4.2. A *proposition* is any $\#$-consistent pair (A_T, A_F) of simple propositions (i.e., $\#$-closed sets). The set of all propositions over $(X, \#)$ will be denoted by $L_f(X, \#) := \{(A_T, A_F): A_T, A_F \in \mathcal{M}(X, \#), A_T(\#)A_F\}$. The set A_T is the *certainly-true* domain of proposition p and the set A_F is the *certainly-false* domain. Sometimes if p is a proposition, we denote by $A_T(p)$ and $A_F(p)$ the certainly-true domain and the certainly-false domain associated with p, respectively.

If $x \in A_T(p)$, then we say that the proposition p is *true* in the state (world) x, while if $x \in A_F(p)$, then the proposition is *false* in this state (world); if neither $x \in A_T(p)$ nor $x \in A_F(p)$, then in the state (world) x the proposition p is neither true nor false, that is, its value is *indeterminate*. The two trivial propositions $\mathbf{0} = (X^{\#}, X)$ and $\mathbf{1} = (X, X^{\#})$ are the *absurd* or *contradictory* proposition and the *certain* or *tautologous* proposition, respectively. A proposition p is *self-contradictory*, or, simply, a *contradiction*, iff $A_T(p) = X^{\#}$.

Remark 1. Note that if p is a proposition, then $A_T(p) \cap A_F(p) \neq \varnothing$ if $\#$ is a paraconsistency or a regular paraconsistency relation, and $A_T(p) \cap A_F(p) = \varnothing$ (self-consistency condition) if $\#$ is a preclusivity relation. In general, $A_T(p) \vee A_F(p) \neq X$.

Theorem 4.2. The set of all propositions has a natural structure $\langle L_f(X, \#), \sqsubseteq, {}^{\perp}, {}^{\sim}, \mathbf{0}, \mathbf{1} \rangle$ with respect to:

1. The partial ordering

$$p \sqsubseteq q \quad \text{iff} \quad A_T(p) \subseteq A_T(q) \quad \text{and} \quad A_F(q) \subseteq A_F(p)$$

2. The fuzzy-like orthocomplementation:

$$(A_T. A_F)^{\perp} = (A_F, A_T)$$

3. The intuitionistic-like orthocomplementation:

$$(A_T, A_F)^{\sim} = (A_F, A_F^{\#})$$

Precisely, if $\#$ is a regular paraconsistency relation, then $L_f(X, \#)$ is a pre-BZ[3] complete lattice; whereas if $\#$ is a preclusivity relation, then $L_f(X, \#)$ is a BZ[3] complete lattice.

$L_f(X, \#)$ is bounded by the minimum element $\mathbf{0} = (X^{\#}, X)$ and the maximum element $\mathbf{1} = (X, X^{\#})$. In the RPQL case $(X^{\#}, X^{\#})$ is one of the possible half propositions; in the MQL case every half proposition coalesces in the unique half proposition $\mathbf{1/2} = (\varnothing, \varnothing)$. The g.l.b. and the l.u.b. of any family of propositions $\{p_j = (A_T^{(j)}, A_F^{(j)}) : j \in J\}$ are given, respectively, by

$$\sqcap(A_T^{(j)}, A_F^{(j)}) = (\bigcap A_T^{(j)}, \bigvee A_F^{(j)}), \qquad \sqcup(A_T^{(j)}, A_F^{(j)}) = (\bigvee A_T^{(j)}, \bigcap A_T^{(j)})$$

The necessity and the possibility of a proposition (A_T, A_F) are, respectively,

$$\Box(A_T, A_F) = (A_T, A_F^{\#}), \qquad \Diamond(A_T, A_F) = (A_F^{\#}, A_F)$$

Trivially, $\Diamond(A_T, A_F) = -(\Box(-(A_T, A_F)))$ and $\Box(A_T, A_F) = -(\Diamond(-(A_T, A_F)))$.

The structure

$$\langle L_f(X, \#), \sqcap, \sqcup, {}^{\perp}, {}^{\sim}, \mathbf{0}, \mathbf{1} \rangle$$

is called the *ortho-pair BZ* (or *fuzzy-instuitionistic*) *propositional lattice* on the frame $(X, \#)$.

The set of all *exact* elements is

$$\begin{aligned} L_e(X, \#) &= \{p \in L_f(X, \#) : p = (p^{\sim})^{\sim}\} \\ &= \{(A, A^{\#}) : A \in \mathcal{M}(X, \#)\} \end{aligned}$$

The restrictions to $L_e(X, \#)$ of the two orthocomplementations defined on $L_f(X, \#)$ coalesce and define a unique orthocomplementation; moreover, the following is a one-to-one correspondence between exact propositions and simple propositions:

$$L_e(X, \#) \equiv \mathcal{M}(X, \#)$$

$$(A, A^{\#}) \leftrightarrow A$$

which allows the identification of the orthocomplemented lattice structures

$$\langle L_e(X, \#), \sqcap, \sqcup, {}^{\perp}, \mathbf{0}, \mathbf{1}\rangle \equiv \langle \mathcal{M}(X, \#), \wedge, \vee, {}^{\#}, X^{\#}, X\rangle$$

For any family of propositions $\{p_j = (A_T^{(j)}, A_F^{(j)}\}$ the proposition $\Pi p_j := (\bigcap A_T^{(j)}, \bigcap A_F^{(j)})$ is well defined and it is called the "product" of these propositions.

4.1. Kripkian Frames and Ortho-Pair BZ Propositional Lattice in Hilbert Space GQM

In the case of GQM based on the Hilbert space $\mathcal{H}$ one can introduce the ϵ certainly-yes domain of an effect $F \in \mathcal{F}(\mathcal{H})$ as the subset of $\mathcal{H}_0$, $M_{1,\epsilon}(F) := \{\psi \in \mathcal{H}_0 : P(\psi, F) \geq 1 - \epsilon\}$, and the ϵ certainly-no domain of the same effect as $M_{0,\epsilon}(F) := \{\varphi \in \mathcal{H}_0 : P(\varphi, F) \leq \epsilon\}$. Trivially, $M_{1,\epsilon}(F)\ (\perp_\epsilon)\ M_{0,\epsilon}(F)$. Let us denote the corresponding $\perp_\epsilon$ closures (which are ϵ simple propositions) as $M_{T,\epsilon}(F) := (M_{1,\epsilon})^{\perp_\epsilon \perp_\epsilon}$ and $M_{F,\epsilon}(F) := (M_{0,\epsilon})^{\perp_\epsilon \perp_\epsilon}$; then $M_{1,\epsilon}(F) \subseteq M_{T,\epsilon}(F)$ and $M_{0,\epsilon}(F) \subseteq M_{F,\epsilon}(F)$, with $M_{T,\epsilon}(F)\ (\perp_\epsilon)\ M_{F,\epsilon}(F)$.

The Hilbert space case can be summarized by the following diagram:

$$\begin{array}{ccc} \text{Effect} & & \text{G-Proposition} \\ F \in \mathcal{F}(\mathcal{H}) & \xrightarrow{\text{ext}_\epsilon} & (M_{1,\epsilon}(F), M_{0,\epsilon}(F)) \\ & & \big\downarrow \text{cl}_\epsilon \\ & & (M_{T,\epsilon}(F), M_{F,\epsilon}(F)) \\ & & \text{BZ-Proposition} \end{array}$$

with respect to which we have that, $\forall \epsilon \in [0, \frac{1}{2})$, $M_{1,\epsilon}(F)\ (\perp_\epsilon)\ M_{0,\epsilon}(F)$ implies $M_{1,\epsilon}(F) \cap M_{0,\epsilon}(F) = \varnothing$ and $\forall \epsilon \in [\frac{1}{2}, 1]$, $M_{i,\epsilon}(F)\ (\perp_\epsilon)\ M_{0,\epsilon}(F)$ implies $M_{1,\epsilon}(F) \cap M_{0,\epsilon}(F) \neq \varnothing$. Preparations from $M_{T,\epsilon}(F) \backslash M_{1,\epsilon}(F)$ [resp.,

$M_{F,\epsilon}(F)\backslash M_{0,\epsilon}(F)]$ can be considered semantical worlds in which statements associated to effect F are ϵ-true [resp., ϵ-false] without the conditions that the probability is $\geq(1-\epsilon)$ [resp. $\leq\epsilon$].

The behavior of Hilbertian propositions associated to effects is given by

$$\begin{array}{cccc} 1 & \mathscr{H} & (\perp_1) & \mathscr{H} \\ \vdots & \vdots & & \vdots \\ \epsilon_2 & M_{T,\epsilon_2}(F) & (\perp_{\epsilon_2}) & M_{F,\epsilon_2}(F) \\ \vee & \cup| & & \cup| \\ \epsilon_1 & M_{T,\epsilon_1}(F) & (\perp_{\epsilon_1}) & M_{F,\epsilon_1}(F) \\ \vee & \cup| & & \cup| \\ \vdots & \vdots & & \vdots \\ 0 & M_1(F) & (\perp_0) & M_0(F) \end{array}$$

5. LANGUAGE ASSOCIATED TO UNSHARP REALIZATION OF HILBERTIAN SHARP OBSERVABLES AND FUZZY-INTUITIONISTIC QUANTUM LOGICS

In Hilbert space quantum mechanics any real-value *observable* $A\in\mathcal{O}$ is (*sharply*) described by a projection-valued (PV) measure on real Borel sets $E_A:\mathscr{B}(\mathbb{R})\mapsto\mathscr{E}(\mathscr{H})$, which, via the spectral theorem, is in a one-to-one correspondence with a self-adjoint densely defined (in general unbounded) operator $\hat{A}:\mathscr{D}_{\hat{A}}\mapsto\mathscr{H}$. An *exact question* is any pair $(A,\Delta)\in\mathcal{O}\times\mathscr{B}(\mathbb{R})$ describing the *elementary statement* of the everyday language: "*a measurement of the physical quantity A gives a value contained in the set* Δ *of real numbers*;" this question is sharply tested by the exact yes–no measurement device $E_A(\Delta)$ (the same yes–no device can test several different exact questions).

Let φ be any preparation and let A be any observable. Let p_φ: $\mathscr{E}(\mathscr{H})\mapsto[0,1]$ be the probability measure (on exact effects) defined for every $E\in\mathscr{E}(\mathscr{H})$ by $p_\varphi(E):=p(\varphi,E)$. The mapping $\mu_{\varphi,A}:\mathscr{B}(\mathbb{R})\mapsto[0,1]$ defined for every $\Delta\in\mathscr{B}(\mathbb{R})$ by

$$\mu_{\varphi,A}(\Delta)=p(\varphi,E_A(\Delta))=\int_{\mathbb{R}}\chi_\Delta\,d(p_\varphi\circ E_A)$$

is a probability measure on real Borel sets, where χ_Δ ($=1$ if $\lambda\in\Delta$ and 0 otherwise) is the characteristic functional of Δ; χ_Δ is interpreted as the *sharp macroscopic localization device* of the "window" that isolates the numerical subset Δ in the reading scale of the apparatus testing the observable A. In particular, via the spectral theorem, the exact effect $E_A(\Delta)$ is implicitly defined by

$$\langle \varphi \mid E_A(\Delta)\varphi \rangle = \|\varphi\|^2 p(\varphi, E_A(\Delta)) = \|\varphi\|^2 \int_{\mathbb{R}} \chi_\Delta \, d(p_\varphi \circ E_A)$$

and the self-adjoint operator $\hat{A}$ by

$$\langle \varphi \mid \hat{A}\varphi \rangle = \|\varphi\|^2 \int_{\mathbb{R}} \mathrm{id} \, d(p_\varphi \circ E_A)$$

A *concrete macroscopic localization device* is any mapping $\omega\colon \mathscr{B}(\mathbb{R}) \times \mathbb{R} \mapsto [0, 1]$ such that for any fixed $x \in \mathbb{R}$, $\omega_x\colon \mathscr{B}(\mathbb{R}) \mapsto [0, 1]$ is a Borel measure and for any fixed $\Delta \in \mathscr{B}(\mathbb{R})$, $\omega_\Delta\colon \mathbb{R} \mapsto [0, 1]$ is a Borel measurable function. An *ω-unsharp realization* of the observable A is the effect-valued (EV) measure $F_A^\omega\colon \mathscr{B}(\mathbb{R}) \mapsto \mathscr{F}(\mathscr{H})$ implicitly defined for every $\Delta \in \mathscr{B}(\mathbb{R})$ by

$$\langle \varphi \mid F_A^\omega(\Delta)\varphi \rangle := \|\varphi\|^2 \int_{\mathbb{R}} \omega_\Delta \, d(p_\varphi \circ E_A)$$

Moreover, for the vectors for which the r.h.s. is defined, a "self-adjoint" operator $\hat{A}^\omega$ is implicitly defined by

$$\langle \varphi \mid \hat{A}^\omega \varphi \rangle = \|\varphi\|^2 \int_{\mathbb{R}} \mathrm{id} \, d(p_\varphi \circ F_A^\omega)$$

A *formal language* for the concrete realization of observables now considered will consist of *elementary sentences* corresponding to *questions* (A, Δ, ω): "*a measurement of the physical quantity A gives a real value in Δ when the latter is realized by the concrete localization device ω.*" Any question (A, Δ, ω) is tested by the corresponding *effect* $F_A^\omega(\Delta)$ and its ϵ-semantics will be characterized by the *ϵ-extension* $\mathrm{ext}_\epsilon(F_A^\omega(\Delta))$, i.e., by the *$\epsilon$-preclusivity proposition* $(M_{T,\epsilon}(F_A^\omega(\Delta)), M_{F,\epsilon}(F_A^\omega(\Delta)))$. Further, our language will allow the use of the connectives "and" (&), "or" ($\underline{o}$), "not" ($\neg$), "impossible" ($\sim$), "necessary" (L), and "possible" (M) to obtain *complex sentences* from the elementary ones. These connectives could be described in the Hilbertian ϵ ortho-pair (fuzzy-intuitionistic, BZ) propositional lattice $L_f(X(\mathscr{H}, \perp_\epsilon))$ by the operations $\sqcap$, $\sqcup$, $'$, $^\sim$, $\Box$, and $\Diamond$, respectively.

All this leads to the conclusion that, from the logical point of view, it is interesting to consider languages equipped with this connectives and realized in terms of several kinds of BZ structures. Analogously to BZ posets, a characteristic feature of these logics, which represent nonstandard versions of quantum logic, is a splitting of the connective "not" into two forms of negation: a *fuzzy-like* negation that gives rise to a *paraconsistent* behavior, and an *intuitionistic-like* negation. We will consider three forms of fuzzy intuitionistic quantum logics: BZL (weak Brouwer–Zadeh), BZL^3 (three-valued Brouwer–Zadeh), and BZL^∞ (infinite-valued Brouwer–Zadeh logic). These logics have a common language, which contains a denumerable set $p_1, \ldots, p_n, \ldots$ of sentential letters and the following primitive connectives: $\neg$ (fuzzy-like negation), $\sim$ (intuitionistic-like negation), and & (conjunction). A privileged sentential letter O will represent the *absurd* assertion. We will use $p, q, r, \ldots$ as metavariables for atomic formulas and $\alpha, \beta, \gamma, \ldots$ as metavariables for formulas. Disjunction ($\underline{\varrho}$) is metatheoretically defined in terms of conjunction and of the fuzzy negation: $\alpha \;\underline{\varrho}\; \beta := \neg(\neg\alpha \;\&\; \neg\beta)$. The modal operators L (necessarily) and M (possibly) are metatheoretically defined in terms of the two negations $L\alpha := \sim \neg\alpha$ and $M\alpha := \neg L \neg\alpha$. Another privileged sentential letter, the *certain* assertion (I), is metatheoretically defied as $I := \neg O$. This language is described by

$$\mathbf{L}_{GQM} := \langle \mathscr{L}_{\mathrm{GQM}}, \&, \underline{\varrho}, \neg, \sim, O, I \rangle$$

6. WEAK BROUWER–ZADEH LOGIC

A natural semantical characterization for the weakest BZL can be given in the framework of an algebraic semantics (from now on, we take into account only BZ structures, disregarding pre-BZ ones).

Definition 6.1. A BZL-algebraic realization is a pair $\mathfrak{M} = \langle \Sigma, v \rangle$, where

$$\langle \Sigma, \wedge, \vee, ', {}^\sim, 0, 1 \rangle$$

is a *BZ* lattice and v is a *valuation mapping* which maps formulas into elements of Σ according to the following conditions:

$$v(\beta \;\&\; \gamma) = v(\beta) \wedge v(\gamma)$$

$$n(\neg\beta) = v(\beta)'$$

$$v(\sim\beta) = v(\beta)^\sim$$

$$v(O) = 0$$

Trivially, from these conditions it follows that

$$v(\beta \;\underline{\varrho}\; \gamma) = v(\beta) \vee v(\gamma)$$

$$v(I) = 1$$

Moreover, $v(L\beta) = \nu(v(\beta))$ and $v(M\beta) = \mu(v(\beta))$.

Definition 6.2. Truth, consequence, logical consequence:

6.2.1. A formula α is *true* in a BZL-realization $\mathfrak{M}$(written $\models_{\mathfrak{M}}\alpha$) iff $v(\alpha) = \mathbf{1}$.

6.2.2. α is a *consequence* in $\mathfrak{M}$ of a set of formulas T ($T \models_{\mathfrak{M}} \alpha$) iff $\forall a \in \Sigma\, [\forall \beta \in T(a \leq v(\beta)$ implies $a \leq v(\alpha))]$.

6.2.3. α is a *BZL logical consequence* of T ($T \models_{\mathrm{BZL}} \alpha$) iff for any BZL-algebraic realization $\mathscr{M}$, $T \models_{\mathfrak{M}} \alpha$.

A Kripke-semantics for BZL was first proposed in Giuntini (1990). A characteristic feature of this semantics is the use of Kripke-frames with two preclusivity relations. One can prove, with standard techniques, that the algebraic and the Kripke-semantics for BZL characterize the same logic. BZL can be axiomatized: a soundness and a completeness theorem are provable, with respect to both semantics (Giuntini, 1990). Moreover, BZL has the finite model property and accordingly is decidable (Giuntini, 1992*b*).

Characteristic logical properties that fail and hold in BZL are the following:

(a) As in *paraconsistent quantum logic* (Dalla Chiara and Giuntini, 1989), the distributive principles, the strong and weak Duns Scoto principles, the noncontradiction, and the excluded middle principles break down for the fuzzy negation.

(b) As in intuitionistic logic, we have

$$\models_{\mathrm{BZL}} \sim(\alpha \;\&\; \sim\alpha); \qquad \not\models_{\mathrm{BZL}} \alpha \,\underline{o}\, \sim\alpha; \qquad \alpha \models_{\mathrm{BZL}} \sim\sim\alpha; \qquad \sim\sim\alpha \not\models_{\mathrm{BZL}} \alpha$$

$$\sim\sim\sim\alpha \models_{\mathrm{BZL}} \sim\alpha; \qquad \text{if } \alpha \models_{\mathrm{BZL}} \beta, \text{ then } \sim\beta \models_{\mathrm{BZL}} \sim\alpha$$

(c) Moreover, we have

$$\sim\alpha \models_{\mathrm{BZL}} \neg\alpha; \qquad \neg\alpha \not\models_{\mathrm{BZL}} \sim\alpha; \qquad \neg\sim\alpha \models_{\mathrm{BZL}} \sim\sim\alpha$$

(d) The modal operators give rise to an S_5-like behavior:

$$L\alpha \models_{\mathrm{BZL}} \alpha; \qquad L(\alpha \;\&\; \beta) \models_{\mathrm{BZL}} L\alpha \;\&\; L\beta; \qquad L\alpha \;\&\; L\beta \models_{\mathrm{BZL}} L(\alpha \;\&\; \beta)$$

$$M(\alpha \;\&\; \beta) \models_{\mathrm{BZL}} M\alpha \;\&\; M\beta; \qquad L\alpha \models_{\mathrm{BZL}} LL\alpha; \qquad M\alpha \models_{\mathrm{BZL}} LM\alpha$$

$$\text{if } \models_{\mathrm{BZL}} \alpha, \text{ then } \models_{\mathrm{BZL}} L\alpha$$

7. A SEMANTICS WITH POSITIVE AND NEGATIVE CERTAINTY DOMAINS

An alternative semantical description for a form of fuzzy intuitionistic logic was first proposed in Cattaneo and Nisticò (1989). The intuitive idea underlying this semantics (founded on the generalization of simple propositions in a Hilbert space outlined in Section 4) can be sketched as follows:

one supposes that interpreting a language means essentially associating to any sentence two *domains of certainty*: the domain of situations where the sentence certainly holds, and the domain of situations where the sentence certainly does not hold. Similarly to Kripke-semantics, the situations we are referring to can be thought of as kinds of possible worlds. However, differently from the standard Kripkean behavior, the positive domain of a given sentence does not generally determine the negative domain of the same sentence. As a consequence, propositions are here identified with particular pairs of sets of worlds, rather than with particular sets of worlds (as happens in the usual possible worlds semantics).

Let us again assume the BZL language. We will define the notion of *realization with positive and negative certainty domains* (briefly, *ortho-pair realization*) for a BZL language.

Definition 7.1. An *ortho-pair realization* is a system $\mathfrak{X} = \langle X, \#, \mathbf{L}, v \rangle$, where:

7.1.1. $(X, \#)$ is a preclusivity space.

According to Section 4, a *simple proposition* is a set of worlds A such that $A = A^{\#\#}$. We recall that a *possible proposition* of the preclusivity space is any pair (A_T, A_F), where A_T, A_F are simple propositions such that $A_T \subseteq A_F^{\#}$. The following operations and relations are defined on the set of all propositions:

The *order-relation*:

$$(A_T, A_F) \sqsubseteq (B_T, B_F) \qquad \text{iff} \quad A_T \subseteq B_T \quad \text{and} \quad B_F \subseteq A_F$$

The *fuzzy complement*:

$$-(A_T, A_F) = (A_F, A_T)$$

The *intuitionistic complement*:

$$\sim(A_T, A_F) = (A_F, A_F^{\#})$$

The *propositional conjunction*:

$$(A_T, A_F) \sqcap (B_T, B_F) = (A_T \cap B_T, A_F \vee B_F)$$

The *propositional disjunction*:

$$(A_T, A_F) \sqcup (B_T, B_F) = (A_T \vee B_T, A_F \cap B_F)$$

The *necessity operator*:

$$\Box(A_T, A_F) = (A_T, A_T^{\#})$$

The *possibility operator*:

$$\Diamond(A_T, A_F) = (A_F^{\#}, A_F)$$

The *impossible proposition*:

$$\mathbf{0} = (\varnothing, X)$$

The *certain proposition*:

$$\mathbf{1} = (X, \varnothing)$$

7.1.2. **L** (the set of the *actual propositions*) is a set of possible propositions from $L_f(X, \#)$ which contains **0** and is closed under $-$, $\sim$, $\sqcap$.

7.1.3. v is a *valuation mapping* that maps formulas into propositions according to the following conditions:

$$v(\beta \,\&\, \gamma) = v(\beta) \sqcap v(\gamma)$$

$$v(\neg \beta) = -v(\beta)$$

$$v(\sim \beta) = \sim v(\beta)$$

$$v(O) = \mathbf{0}$$

The other basic semantical definitions are as in the algebraic semantics.

One can show that in any ortho-pair realization the set of propositions **L** gives rise to a BZ lattice (see Section 4). As a consequence, one immediately proves a soundness theorem with respect to the ortho-pair semantics. One might guess that the ortho-pair semantics characterizes the logic BZL. However, this conjecture has a negative answer. As a counterexample, let us consider an instance of the fuzzy excluded middle and an instance of the intuitionistic excluded middle applied to the same formula α:

$$\alpha \,\underline{\varrho}\, \neg\alpha \qquad \text{and} \qquad \alpha \,\underline{\varrho}\, \sim\alpha$$

One can easily check that they are logically equivalent in the ortho-pair semantics. Indeed, for any ortho-pair realization $\mathfrak{X}$:

$$\alpha \,\underline{\varrho}\, \neg\alpha \models_{\mathfrak{X}} \alpha \,\underline{\varrho}\, \sim\alpha \qquad \text{and} \qquad \alpha \,\underline{\varrho}\, \sim\alpha \models_{\mathfrak{X}} \alpha \,\underline{\varrho}\, \neg\alpha$$

However, generally,

$$\alpha \,\underline{\varrho}\, \neg\alpha \not\models_{\mathrm{BZL}} \alpha \,\underline{\varrho}\, \sim\alpha$$

For instance, let us consider the following BZL-realization $\mathfrak{M} = \langle \Sigma, v \rangle$, where the support Σ of $\mathfrak{M}$ is the real interval [0, 1] equipped with the usual order of real numbers and algebraic structure defined as follows:

$$a' = 1 - a$$

$$a^{\sim} = \begin{cases} 1 & \text{if } a = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbf{1} = 1; \qquad \mathbf{0} = 0$$

Let $0 < v(p) < 1/2$. We will have $v(p \vee \sim p) = \max(v(p), 0) = v(p) < 1/2$. But $v(p \vee \neg p) = \max(v(p), 1 - v(p)) = 1 - v(p) \geq 1/2$. Hence: $v(p \vee \sim p) < v(p \vee \neg p)$.

As a consequence, the ortho-pair-semantics characterizes a logic stronger than BZL. We will call this logic BZL^3. BZL^3 can be axiomatized and a completeness theorem can be proved with respect to the ortho-pair semantics (Giuntini, 1990). BZL^3 can be equivalently characterized by means of an algebraic semantics based on the class of all BZ^3 lattices.

8. INFINITE-VALUED BROUWER–ZADEH LOGIC

In the ortho-pairs semantics, a proposition (A_T, A_F) of an ortho-pair realization $\mathfrak{X} = \langle X, \#, \mathbf{L}, v \rangle$ can be thought of as a mapping π: $\{0, 1/2, 1\} \to \mathscr{P}(X)$ [where $\mathscr{P}(X)$ is the power set of X] which satisfies the following conditions:

(i) $\pi_1 = A_T$.
(ii) $\pi_0 = A_F$.
(iii) $\pi_{1/2} = X \backslash \{A_T \cup A_F\}$.

Accordingly, the ortho-pair semantics admits of a natural infinite generalization by replacing the set $\{0, 1/2, 1\}$ with the interval $[0, 1]$.

Definition 8.1. An *ortho-infinite many-valued realization* is a system $\mathfrak{X} = \langle X, \#, \mathbf{L}, v \rangle$, where

8.1.1. $(X, \#)$ is a preclusivity space. A *possible proposition* of the preclusivity space is a mapping π: $[0, 1] \mapsto \mathscr{P}(X)$, $r \to \pi_r$, which satisfies the following conditions:

(a) π_1 and π_0 are simple propositions of the preclusivity space.
(b) $\pi_1 \subseteq (\pi_0)^{\#}$.
(c) $\pi_r \cap \pi_s = \varnothing$, if $r \neq s$.
(d) $\forall x \in X, \exists r \in [0, 1]$ such that $x \in \pi_r$.

The following relations and operations can be defined on the set of all possible propositions, whatever be the possible propositions π, ρ: $[0, 1] \mapsto \mathscr{P}(X)$:

The *order relation*:

$$\pi \sqsubseteq \rho \quad \text{if} \quad \forall x \in X\colon \ x(\pi) \leq x(\rho), \qquad \text{where} \quad x(\pi) = r \quad \text{iff} \quad x \in \pi_r$$

The *fuzzy-negation*:

$$(\pi')_r = \{x \mid x(\pi) = 1 - r\}$$

The *intuitionistic-negation*:

$$(\pi^{\sim})_1 = \pi_0$$

$$(\pi^{\sim})_0 = (\pi_0)^{\#}$$

$$(\pi^{\sim})_r = (\pi')_r \setminus (\pi_0)^{\#} \qquad \text{if} \quad r \notin \{0, 1\}$$

The *propositional conjunction*

$$(\pi \sqcap \rho)_1 = \pi_1 \wedge \rho_1$$

$$(\pi \sqcap \rho)_0 = \pi_0 \vee \rho_1$$

$$(\pi \sqcap \rho)_r = \{x \mid \exists a, b\ (x \in \pi_a,\ x \in \rho_b, \text{ and } r = \min(a, b))\} \setminus \{\pi_0 \vee \rho_0\}, \quad \text{if} \quad r \notin \{0, 1\}$$

The *propositional disjunction*:

$$(\pi \sqcup \rho)_1 = \pi_1 \vee \rho_1$$

$$(\pi \sqcup \rho)_0 = \pi_0 \wedge \rho_1$$

$$(\pi \sqcup \rho)_r = \{x \mid \exists a, b\ (x \in \pi_a,\ x \in \rho_b, \text{ and } r = \max(a, b))\} \setminus \{\pi_1 \vee \rho_1\} \quad r \notin \{0, 1\}$$

The *necessity operation*:

$$(\nu(\pi))_1 = \pi_1$$

$$(\nu(\pi))_0 = (\pi_1)^{\#}$$

$$(\nu(\pi))_r = \{x \mid x(\pi) = r\} \setminus (\pi_1)^{\#} \qquad \text{if} \quad r \notin \{0, 1\}$$

The *possibility operation*:

$$(\mu(\pi))_1 = \{\pi_0\}^{\#}$$

$$(\mu(\pi))_0 = (\pi_0)$$

$$(\mu(\pi))_r = \pi_r \setminus ((\pi_0)^{\#} \cup \pi_0) \qquad \text{if } r \notin \{0, 1\}$$

The *impossible proposition*:

$$(\mathbf{0})_0 = X$$

$$(\mathbf{0})_r = \varnothing \qquad \text{if} \quad r \neq 0$$

The *certain proposition*:

$$\mathbf{1} = \mathbf{0}'$$

In the particular case of the orthoframe $(\mathscr{H}_0, \perp)$ based on the Hilbert space

$\mathcal{H}$, any effect $F \in \mathcal{F}(\mathcal{H})$ gives rise to a possible proposition π^F: $[0, 1] \mapsto \mathcal{P}(\mathcal{H}_0)$ which associates to any $r \in [0, 1]$ the subset of $\mathcal{H}_0$.

$$\pi_r^F := \{\psi \in \mathcal{H}_0 \colon p(\psi, F) = r\} \equiv \{\psi \in \mathcal{H} \colon \langle \psi \mid F\psi \rangle = r\|\psi\|^2\}$$

8.1.2. **L** is defined as in the ortho-pair semantics.

One can show that the structure $\langle \mathbf{L}, \sqcap, \sqcup, ', {}^{\sim}, \mathbf{1}, \mathbf{0} \rangle$ is a BZ lattice.

8.1.3. v is defined as in the ortho-pair semantics.

Let BZL^∞ be the logic which is characterized by the class of all ortho-infinite many-valued realizations. One can prove that

$$\mathrm{BZL} \subset \mathrm{BZL}^\infty \subset \mathrm{BZL}^3$$

with respect to the relation of *logical consequence*. If we take into account only the notion of *logical truth*, then one can prove that

$$\mathrm{BZL} \subset \mathrm{BZL}^\infty \qquad \text{but} \qquad \mathrm{BZL}^\infty = \mathrm{BZL}^3$$

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